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TOPOLOGIES LARGER THAN α-COMPACT TOPOLOGIES

Robert L. BLAIR, Lech T. POLKOWSKI and Mary Anne SWARDSON
Department of Mathematics, 321 Morton Hall, Ohio University, Athens, OH 45701, USA

Dedicated to the memory of Eric K. van Douwen

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Let $\langle X, \mathcal{T}\rangle$ be an α-compact space in the sense of Herrlich and let $\mathcal{T}'$ be a Tychonoff topology on X such that $\mathcal{T} \subset \mathcal{T}'$. Conditions for α-compactness of $\mathcal{T}'$ are obtained which continue work of Comfort, Retta, Williams, and others. Studies of Kato, Van Douwen, Williams, and others are also continued by obtaining sufficient conditions for α-compactness of partial box products of families of α-compact spaces.

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α-compact space	zero-dimensional space	P_κ-space
realcompact space	completely uniformizable space	G_κ-modification
measurable cardinal	α-completely uniformizable space	κ-box product

1. Introduction

By a "topological space" (or simply a "space"), we shall always mean a Tychonoff space, and α, κ, λ, and μ will always denote infinite cardinals.

If $\langle X, \mathcal{T}\rangle$ is a space, then $\mathcal{T}(\kappa)$ denotes the G_κ-modification of $\mathcal{T}$, i.e., the topology on X having as a base the collection $\{\bigcap \mathcal{G}: \mathcal{G} \subset \mathcal{T}$ and $0 \neq |\mathcal{G}| < \kappa\}$. It is clear that $\mathcal{T}(\omega) = \mathcal{T}$, and it is easily verified that $\mathcal{T}(\kappa)$ is zero-dimensional (i.e., has a base consisting of clopen sets) if $\kappa > \omega$. Thus, in any case, $\langle X, \mathcal{T}(\kappa)\rangle$ is again Tychonoff.

In this paper we study α-compact spaces in the sense of Herrlich (see Section 2 for definitions) and, in particular, the question of when a topology larger than an α-compact topology is again α-compact. A few results concerning this question are already known. We describe these, as well as our present contributions, as follows:

Let $\langle X, \mathcal{T}\rangle$ be an α-compact space, let $\omega \leq \kappa \leq m(\alpha)$ (where $m(\alpha)$ is the smallest measurable cardinal $\geq \alpha$), and assume that $\mathcal{T} \subset \mathcal{T}' \subset \mathcal{T}(\kappa)$. In [5, 6.1], Comfort and Retta show that $\langle X, \mathcal{T}'\rangle$ is α-compact if (1) either $\kappa \leq \alpha$ or $\mathcal{T}' = \mathcal{T}(m(\alpha))$, and in [17, Cor. 4] Retta asserts, but with an incorrect proof, that $\langle X, \mathcal{T}'\rangle$ is α-compact if (2) $\mathcal{T}' = \mathcal{T}(\kappa)$ (see Remark 3.2(e) below). Moreover, in [20, 3.3], Williams shows that $\langle X, \mathcal{T}'\rangle$ is α-compact if (3) $\langle X, \mathcal{T}'\rangle$ is zero-dimensional and $\mathcal{T}(\mu) \subset \mathcal{T}' \subset \mathcal{T}(\mu^+)$, where $\mu^+ \leq m(\alpha)$ and $\alpha = \omega_1$ (see Remark 3.2(c)).

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In our main result, Theorem 3.1, we give a new proof of case (1) above; a correct proof of case (2); a proof of case (3) without the hypothesis of zero-dimensionality, and for arbitrary α; and, finally, a proof of α-compactness of $\langle X, \mathcal{T}'\rangle$ for the case in which $\langle X, \mathcal{T}'\rangle$ is a P_κ-space.

Again let $\omega \leq \kappa \leq m(\alpha)$, and let $\langle X, \mathcal{T}\rangle$ and $\langle X, \mathcal{T}'\rangle$ be the Tychonoff and κ-box products, respectively, of a family of α-compact spaces. (Then $\langle X, \mathcal{T}\rangle$ is α-compact [11, 3.4], and clearly $\mathcal{T} \subset \mathcal{T}' \subset \mathcal{T}(\kappa)$.) In [13, 2.4], Kato shows that $\langle X, \mathcal{T}'\rangle$ is α-compact for the case $\alpha = \omega_1$ (i.e., for the realcompact case), and the same result has been obtained independently by Eric van Douwen (unpublished). Whether $\langle X, \mathcal{T}'\rangle$ is α-compact for arbitrary α remains an open question. (Retta claims an affirmative answer in [17, Cor. 5], but again his proof is incorrect.) Here we show, among other things, that $\langle X, \mathcal{T}'\rangle$ is α-compact if each factor space is zero-dimensional (Theorem 4.4) or if each factor space is completely uniformizable (Corollary 4.7).

Finally, we note that it is an easy exercise, based on the techniques of [9, 8.18], to show that if a space X is hereditarily α-compact, then so is X with *any* larger Tychonoff topology.

2. Definitions and preliminaries

Let X be a space. A *zero-set* of X is a set of the form $f^{-1}(\{0\})$, where f is a real-valued continuous function on X. A *cozero-set* of X is the complement of a zero-set of X. If $\mathcal{S}$ is a collection of subsets of X, then $\mathcal{S}$ has the *α-intersection property* if $\bigcap \mathcal{A} \neq \emptyset$ whenever $\mathcal{A} \subset \mathcal{S}$ with $0 \neq |\mathcal{A}| < \alpha$, and $\mathcal{S}$ is *fixed* (resp. *free*) if $\bigcap \mathcal{S} \neq \emptyset$ (resp. $\bigcap \mathcal{S} = \emptyset$). Moreover, if $A \subset X$, we set $\mathcal{S}|A = \{S \cap A \colon S \in \mathcal{S}\}$.

A *z-filter* on X is a filter of zero-sets of X; a *prime* z-filter on X is a z-filter $\mathcal{F}$ on X with the property that if Z_1 and Z_2 are zero-sets of X with $Z_1 \cup Z_2 \in \mathcal{F}$, then $Z_1 \in \mathcal{F}$ or $Z_2 \in \mathcal{F}$. A *z-ultrafilter* on X is a maximal z-filter on X. Clearly every z-ultrafilter is prime.

We assume familiarity with the basic theory of z-filters, prime z-filters, and z-ultrafilters on X (see [9]). In particular, we recall that if $f: X \to Y$ is continuous and if $\mathcal{F}$ is a z-filter on X, then $f^{\#}(\mathcal{F})$ denotes the z-filter on Y consisting of all zero-sets Z of Y such that $f^{-1}(Z) \in \mathcal{F}$ [9, 4.12].

2.1. Proposition. *If $f: X \to Y$ is continuous, if $\alpha > \omega$, and if $\mathcal{F}$ is a z-ultrafilter on X with the α-intersection property, then $f^{\#}(\mathcal{F})$ is a z-ultrafilter on Y with the α-intersection property.*

Proof. By [9, 4.12], $f^{\#}(\mathcal{F})$ is prime, and clearly $f^{\#}(\mathcal{F})$ has the α-intersection property. Moreover, since $\alpha > \omega$, $f^{\#}(\mathcal{F})$ is closed under countable intersection and hence (by [1, 2.2]) is a z-ultrafilter on Y. $\square$

A space X is *α-compact* (in the sense of Herrlich [11]) if every z-ultrafilter on X with the α-intersection property is fixed. Thus X is compact (resp. realcompact) if and only if X is ω-compact (resp. ω_1-compact). See [5] for a recent survey of results on α-compactness as well as for additional references.

A cardinal m is *measurable* if there exists a free ultrafilter on the set m with the m-intersection property (see e.g. [4, p. 186]). Thus, in particular, ω is measurable. (It is known to be consistent with ZFC that there are no uncountable measurable cardinals, but it is unknown whether ZFC implies that there are not any.) We denote by $m(\alpha)$ the smallest measurable cardinal m (if it exists) such that $\alpha \leqslant m$.

As will be clear in Section 3, measurable cardinals play an essential role in the study of preservation of α-compactness by larger topologies. The main tool in this study is the following lemma (which generalizes [5, 5.2]):

2.2. Lemma. *Let $\mathscr{F}$ be a z-ultrafilter on a space X and let $\Phi = \{\mathscr{A}\colon \mathscr{A} \subset \mathscr{F}, \bigcap \mathscr{A} = \emptyset$, and $\bigcap \mathscr{A}'$ is clopen in X for every $\mathscr{A}' \subset \mathscr{A}$ with $|\mathscr{A}'| < |\mathscr{A}|\}$. If $\Phi \neq \emptyset$, then* $\min\{|\mathscr{A}|\colon \mathscr{A} \in \Phi\}$ *is a measurable cardinal.*

Proof. Let $m = \min\{|\mathscr{A}|\colon \mathscr{A} \in \Phi\}$, choose $\mathscr{A} \in \Phi$ with $|\mathscr{A}| = m$, and write $\mathscr{A} = \{A_\xi\colon \xi \in m\}$. Let $B_0 = X$ and, for $0 < \xi < m$, let $B_\xi = \bigcap_{\eta<\xi} \mathscr{A}_\eta$. Then $\bigcap_{\xi\in m} B_\xi = \emptyset$ and, as is readily verified, for every $x \in X$, $\min\{\zeta \in m\colon x \notin B_\zeta\}$ is a successor ordinal. We can therefore define a function $f\colon X \to m$ (where m has its discrete topology) as follows: $f(x) = \xi$ if and only if $x \in B_\xi - B_{\xi+1}$. Since $f^{-1}(\{\xi\}) = B_\xi - B_{\xi+1}$ and B_ξ and $B_{\xi+1}$ are both clopen in X, f is continuous. Then $f^{\#}(\mathscr{F})$ is a prime filter on m [9, 4.12], and hence an ultrafilter on m.

Suppose there exists $\xi \in \bigcap f^{\#}(\mathscr{F})$. Then $\{\xi\} \in f^{\#}(\mathscr{F})$, and hence $B_\xi - B_{\xi+1} \in \mathscr{F}$. Since $B_\xi - B_{\xi+1}$ is clopen in X, it follows easily that $\mathscr{A}^* = \{A_\eta\colon \eta \leqslant \xi\} \cup \{B_\xi - B_{\xi+1}\}$ is in Φ. But clearly $m \geqslant \omega$ and hence $|\mathscr{A}^*| < m$, contrary to the minimality of m. Thus $f^{\#}(\mathscr{F})$ is free.

Next let $\mathscr{E} \subset f^{\#}(\mathscr{F})$ with $|\mathscr{E}| < m$, and let $\mathscr{B} = \{f^{-1}(E)\colon E \in \mathscr{E}\}$. Then $\mathscr{B} \subset \mathscr{F}$ and $\bigcap \mathscr{B}'$ is clopen in X for every $\mathscr{B}' \subset \mathscr{B}$. Hence $\bigcap \mathscr{B} \neq \emptyset$ by the minimality of m, so $\bigcap \mathscr{E} \neq \emptyset$, and we conclude that $f^{\#}(\mathscr{F})$ has the m-intersection property. $\square$

2.3. Remarks. (a) In the statement of Lemma 2.2, the phrase "is clopen in X" cannot be replaced by "is a zero-set in X". To see this, let $\mathscr{F}$ be the (unique) free z-ultrafilter on the space ω_1 of countable ordinals, let $\Phi' = \{\mathscr{A}\colon \mathscr{A} \subset \mathscr{F}, \bigcap \mathscr{A} = \emptyset$, and $\bigcap \mathscr{A}'$ is a zero-set in ω_1 for every $\mathscr{A}' \subset \mathscr{A}$ with $|\mathscr{A}'| < |\mathscr{A}|\}$, and let $\mathscr{B} = \{[\xi, \omega_1)\colon \xi < \omega_1\}$. Then $\mathscr{B} \in \Phi'$ (in fact, as is readily verified, $\bigcap \mathscr{B}'$ is a zero-set in ω_1 for every $\mathscr{B}' \subset \mathscr{B}$) and we have $\omega_1 \leqslant m = \min\{|\mathscr{A}|\colon \mathscr{A} \in \mathscr{F}\} < m(\omega_1)$. Hence m is not measurable.

(b) Let $\mathscr{F}$ and Φ be as in Lemma 2.2 and let $\Phi^* = \{\mathscr{A}\colon \mathscr{A} \subset \mathscr{F}, \bigcap \mathscr{A} = \emptyset$, and $\bigcap \mathscr{A}'$ is clopen in X for every $\mathscr{A}' \subset \mathscr{A}\}$. Then it is not difficult to show that $\Phi \neq \emptyset$ if and only if $\Phi^* \neq \emptyset$, and that, in this case, $\min\{|\mathscr{A}|\colon \mathscr{A} \in \Phi\} = \min\{|\mathscr{A}|\colon \mathscr{A} \in \Phi^*\}$.

A space $\langle X, \mathscr{T}\rangle$ is a *P_κ-space* if $\bigcap \mathscr{G} \in \mathscr{T}$ whenever $\mathscr{G} \subset \mathscr{T}$ with $0 \neq |\mathscr{G}| < \kappa$.

3. The main theorem

The following is our principal theorem. For completeness, we include some known cases in its statement and proof.

3.1. Theorem. *Let $\langle X, \mathcal{T}\rangle$ be an α-compact space and let $\mathcal{T}'$ be a Tychonoff topology on X such that $\mathcal{T} \subset \mathcal{T}' \subset \mathcal{T}(\kappa)$, where $\omega \leqslant \kappa \leqslant m(\alpha)$. If any one of the following conditions is satisfied, then $\langle X, \mathcal{T}'\rangle$ is α-compact.*

(1) $\kappa \leqslant \alpha$.

(2) *$\langle X, \mathcal{T}'\rangle$ is a P_κ-space.*

(3) *$\kappa = \mu^+$ and $\mathcal{T}(\mu) \subset \mathcal{T}'$ (hence $\mathcal{T}(\mu) \subset \mathcal{T}' \subset \mathcal{T}(\mu^+)$).*

(4) $\mathcal{T}' = \mathcal{T}(\kappa)$.

Proof. We may obviously assume that $\alpha > \omega$.

Let $\mathcal{F}$ be a z-ultrafilter on $\langle X, \mathcal{T}'\rangle$ with the α-intersection property, and suppose that $\bigcap \mathcal{F} = \emptyset$. Let f be the identity map from $\langle X, \mathcal{T}'\rangle$ onto $\langle X, \mathcal{T}\rangle$. Since f is continuous, $f^{\#}(\mathcal{F})$ is a z-ultrafilter on $\langle X, \mathcal{T}\rangle$ with the α-intersection property (Proposition 2.1) and hence there exists $x \in \bigcap f^{\#}(\mathcal{F})$. Since $x \notin \bigcap \mathcal{F}$, there exists a cozero-set neighborhood P of x in $\langle X, \mathcal{T}'\rangle$ with $X - P \in \mathcal{F}$, and there exists a zero-set neighborhood Z of x in $\langle X, \mathcal{T}'\rangle$ with $Z \subset P$. Let $Q = X - Z$, and note that Q meets every member of $\mathcal{F}$. Since Q is a cozero-set, and hence z-embedded, in $\langle X, \mathcal{T}'\rangle$, it follows from [2, 3.1] that $\mathcal{F}|Q$ is a z-ultrafilter on $\langle Q, \mathcal{T}'|Q\rangle$. Moreover, it is clear that $\mathcal{F}|Q$ has the α-intersection property.

Observe next that since $\mathcal{T}' \subset \mathcal{T}(\kappa)$, we have $\operatorname{int}_{\mathcal{T}'} Z \in \mathcal{T}(\kappa)$. Hence there exists $\mathcal{G} \subset \mathcal{T}$ with $|\mathcal{G}| < \kappa$ such that $x \in \bigcap \mathcal{G} \subset \operatorname{int}_{\mathcal{T}'} Z$, and for every $G \in \mathcal{G}$ there exists a zero-set neighborhood Z_G of x in $\langle X, \mathcal{T}\rangle$ such that $x \in Z_G \subset G$. Let $\mathcal{B} = \{Z_G \cap Q \colon G \in \mathcal{G}\}$ and note that $|\mathcal{B}| < \kappa$.

We first verify the following:

(i) $\mathcal{B} \subset \mathcal{F}|Q$.

(ii) $\bigcap \mathcal{B} = \emptyset$.

To prove (i), we need only note that since $x \in \bigcap f^{\#}(\mathcal{F})$, we have $Z_G \in f^{\#}(\mathcal{F}) \subset \mathcal{F}$ for every $G \in \mathcal{G}$. For (ii), we have $\bigcap \mathcal{B} = (\bigcap_{G \in \mathcal{G}} Z_G) \cap Q \subset (\bigcap \mathcal{G}) \cap Q \subset Z \cap Q = \emptyset$.

To finish the proof for Case 1 ($\kappa \leqslant \alpha$), note simply that by (i) and (ii) we have $\alpha \leqslant |\mathcal{B}| < \kappa \leqslant \alpha$, a contradiction.

We may therefore assume, for the rest of the proof, that $\alpha < \kappa$.

Let $\Phi = \{\mathcal{A} \colon \mathcal{A} \subset \mathcal{F}|Q, \bigcap \mathcal{A} = \emptyset$, and $\bigcap \mathcal{A}'$ is clopen in $\langle Q, \mathcal{T}'|Q\rangle$ for every $\mathcal{A}' \subset \mathcal{A}$ with $|\mathcal{A}'| < |\mathcal{A}|\}$. To complete the proof, it will suffice to show, in each of the remaining three cases, that $\mathcal{B} \in \Phi$. (For then $m = \min\{|\mathcal{A}| \colon \mathcal{A} \in \Phi\}$ is measurable by Lemma 2.2, and since clearly $\alpha \leqslant m$, we then have $m(\alpha) \leqslant m \leqslant |\mathcal{B}| < \kappa$, a contradiction.)

Now (in view of (i) and (ii)) to show that $\mathcal{B} \in \Phi$, we need only prove the following:

(iii) $\bigcap \mathcal{B}'$ is clopen in $\langle Q, \mathcal{T}'|Q\rangle$ for every $\mathcal{B}' \subset \mathcal{B}$ with $|\mathcal{B}'| < |\mathcal{B}|$.

Assume then that $\mathscr{B}' \subset \mathscr{B}$ with $|\mathscr{B}'| < |\mathscr{B}|$. Then there exists $\mathscr{G}' \subset \mathscr{G}$ with $|\mathscr{G}'| = |\mathscr{B}'|$ such that $\mathscr{B}' = \{Z_G \cap Q: G \in \mathscr{G}'\}$. Let $\mathscr{Z} = \{Z_G: G \in \mathscr{G}'\}$. Since $\bigcap \mathscr{B}' = (\bigcap \mathscr{Z}) \cap Q$, it suffices now to show that $\bigcap \mathscr{Z} \in \mathscr{T}'$.

For each $G \in \mathscr{G}'$ there exists $\mathscr{H}_G \subset \mathscr{T}$ such that $|\mathscr{H}_G| \leq \omega$ and $Z_G = \bigcap \mathscr{H}_G$.

Case (2). We assume that $\langle X, \mathscr{T}' \rangle$ is a P_κ-space. For each $G \in \mathscr{G}'$ we have $\mathscr{H}_G \subset \mathscr{T}'$ and $|\mathscr{H}_G| \leq \alpha < \kappa$, so $Z_G \in \mathscr{T}'$. Then, since $|\mathscr{Z}| \leq |\mathscr{G}| < \kappa$, we have $\bigcap \mathscr{Z} \in \mathscr{T}'$.

Case (3). We assume that $\kappa = \mu^+$ and $\mathscr{T}(\mu) \subset \mathscr{T}'$. Since $|\mathscr{B}| < \kappa$ and $\alpha < \kappa$, we have $|\mathscr{G}'| = |\mathscr{B}'| < |\mathscr{B}| \leq \mu$ and $\omega < \alpha \leq \mu$, and hence $|\bigcup_{G \in \mathscr{G}'} \mathscr{H}_G| \leq |\mathscr{G}'| \cdot \omega < \mu$. Since $\bigcap \mathscr{Z} = \bigcap (\bigcup_{G \in \mathscr{G}'} \mathscr{H}_G)$, it follows that $\bigcap \mathscr{Z} \in \mathscr{T}(\mu) \subset \mathscr{T}'$.

Case (4). Finally, we assume that $\mathscr{T}' = \mathscr{T}(\kappa)$. Since $|\mathscr{G}'| < \kappa$ and $\omega < \alpha < \kappa$, we have $|\bigcup_{G \in \mathscr{G}'} \mathscr{H}_G| < \kappa$, and hence $\bigcap \mathscr{Z} \in \mathscr{T}(\kappa) = \mathscr{T}'$. $\square$

3.2. Remarks. (a) Case (1) of Theorem 3.1 is due to Comfort and Retta [5, 6.1]. For another proof of this case when $\kappa = \alpha = \omega_1$, see Hernández [10, Cor. 7]. The instance $\mathscr{T}' = \mathscr{T}(\kappa)$ and $\kappa = \alpha = \omega_1$ of Case (1) is due to Frolík [8, Theorem 4]. See also Wheeler [18, 5.2] and the remarks of [5, 4.8(a)].

(b) Case (2) of Theorem 3.1 is new. We note that this case cannot be replaced by the hypothesis "$\langle X, \mathscr{T}' \rangle$ is zero-dimensional." To see this, let X be the ordinal space $\omega_1 + 1$ with its order topology $\mathscr{T}$, and let $\mathscr{T}'$ be the topology on X generated by $\mathscr{T} \cup \{\{\omega_1\}\}$. Then $\langle X, \mathscr{T} \rangle$ is compact, and hence realcompact ($=\omega_1$-compact), and clearly $\mathscr{T} \subset \mathscr{T}' \subset \mathscr{T}(m(\omega_1))$. Moreover, since $\mathscr{T}$ is zero-dimensional and $\{\omega_1\}$ is $\mathscr{T}'$-clopen, $\mathscr{T}'$ is also zero-dimensional. But ω_1 is $\mathscr{T}'$-closed in X and $\langle \omega_1, \mathscr{T}' | \omega_1 \rangle = \langle \omega_1, \mathscr{T} | \omega_1 \rangle$ is not realcompact, and thus $\langle X, \mathscr{T}' \rangle$ is not realcompact. (The space $\langle X, \mathscr{T}' \rangle$ is described in [5, 6.5].)

(c) Case (3), for $\alpha = \omega_1$, was obtained by Williams [20, 3.3], but with the unnecessary hypothesis that $\mathscr{T}'$ be zero-dimensional. (Williams informs us that the hypothesis "$\mathscr{T} \subset \mathscr{T}' \subset \mathscr{T}(\alpha^+)$" of [20, 3.3(2)] is a typographical error, and that it should read "$\mathscr{T}(\alpha) \subset \mathscr{T}' \subset \mathscr{T}(\alpha^+)$". As actually stated, alternative (2) of [20, 3.3] is false, even when $\mathscr{T}'$ is zero-dimensional.) We note that the hypothesis $\mathscr{T}(\mu) \subset \mathscr{T}' \subset \mathscr{T}(\mu^+)$ does not necessarily imply that $\mathscr{T}'$ is zero-dimensional (even for $\mu > \omega$). In fact, the authors show in [3] that if μ is regular, then there is a space $\langle X, \mathscr{T} \rangle$ (which is α-compact if $\mu < m(\alpha)$) and a Tychonoff topology $\mathscr{T}'$ on X such that $\mathscr{T}(\mu) \subset \mathscr{T}' \subset \mathscr{T}(\mu^+)$ and such that $\mathscr{T}'$ is not zero-dimensional at any point of X.

(d) The instance $\kappa = m(\alpha)$ of Case (4) was proved by Comfort and Retta [5, 6.1] and, for $\alpha = \omega_1$, by Williams [20, 3.2]. It also follows from our Case 2 (since $\langle X, \mathscr{T}(m(\alpha)) \rangle$ is a $P_{m(\alpha)}$-space [5, 2.5]). The instance $\kappa < m(\alpha)$ of Case (4) is, of course, an immediate consequence of our Case (3).

(e) Case (4), for $\alpha = \omega_1$, was proved by Kato [13, 3.1] (by an entirely different method). Case (4), for arbitrary α, is stated by Retta as Corollary 4 in [17], but Retta's proof is incorrect since it depends on Theorems 1 and 2 of [17], both of which are false. (The z-ultrafilter $\mathscr{F}$ of 2.3 above provides a counterexample to [17, Theorem 1], and easy examples show that [17, Theorem 2] is also false. The error in [17, Theorem 1] has been observed independently by Retta (unpublished).)

(f) We note that $m(\alpha)$ is an upper bound for the set of cardinals κ for which $\langle X, \mathcal{T}(\kappa)\rangle$ is α-compact whenever a space $\langle X, \mathcal{T}\rangle$ is α-compact. Indeed, for every α there is a compact (hence α-compact) space $\langle X, \mathcal{T}\rangle$ such that, for every $\kappa > m(\alpha)$, $\langle X, \mathcal{T}(\kappa)\rangle$ is not α-compact. To see this, we need only take X to be the one-point compactification of a discrete space of cardinality $m(\alpha)$. Then, for $\kappa > m(\alpha)$, it is clear that $\langle X, \mathcal{T}(\kappa)\rangle$ is discrete. Since $|X| = m(\alpha)$, it follows from [5, 5.3] that $\langle X, \mathcal{T}(\kappa)\rangle$ is not α-compact.

4. The box product problem

Let $\langle X_i: i \in I\rangle$ be a family of spaces and κ an infinite cardinal. The *κ-box product* of $\langle X_i: i \in I\rangle$ (denoted by $\kappa - \Box_{i\in I} X_i$) is the space with the Cartesian product $\prod_{i\in I} X_i$ as its underlying set, and which has, as a base, sets of the form $\bigcap_{i\in J} \pi_i^{-1}(U_i)$, where $J \subset I$ with $0 \neq |J| < \kappa$, π_i is the ith projection map, and U_i is open in X_i for all $i \in J$. (Thus $\omega - \Box_{i\in I} X_i$ is $\prod_{i\in I} X_i$ with its usual (Tychonoff) product topology.) The $|I|^+$-box product of $\langle X_i: i \in I\rangle$ is denoted simply by $\Box_{i\in I} X_i$. A base for the latter consists of all sets of the form $\prod_{i\in I} U_i$, where each U_i is open in X_i.

We are interested here in the following question:

4.1. Question. If $\langle X_i: i \in I\rangle$ is a family of α-compact spaces, and if $\omega \leqslant \kappa \leqslant m(\alpha)$, is $\kappa - \Box_{i\in I} X_i$ also α-compact?

It is known that Question 4.1 can be answered affirmatively in several cases. We summarize these already known results in the following remarks.

4.2. Remarks. (a) The answer to Question 4.1 is "yes" if $\kappa = \omega$. (For $\kappa = \omega = \alpha$, this is just the Tychonoff product theorem (see e.g. [9, 6.8]). For $\kappa = \omega$ and $\alpha \geqslant \omega_1$, see [9, 8.11] and [11, 3.4].)

(b) The answer to Question 4.1 is shown by Comfort and Retta [5, 4.7(c)] to be "yes" if $\kappa \leqslant \alpha$. (This follows, as in [5], from Case (1) of Theorem 3.1 since $\mathcal{T} \subset \mathcal{T}' \subset \mathcal{T}(\kappa)$, where $\mathcal{T}$ and $\mathcal{T}'$ are the Tychonoff and κ-box product topologies, respectively, of $\langle X_i: i \in I\rangle$.) For the special case in which $\kappa = \alpha = \omega_1$ and each X_i is a P_{ω_1}-space, see [15, 4.8].

(c) The answer to Question 4.1 is "yes" if $\alpha = \omega_1$. This is due, independently, to Kato [13, 2.4] and Van Douwen (unpublished). See also Williams [19, 1.15].

Question 4.1, in general, remains open. (In [17, Cor. 5], Retta asserts an affirmative answer to Question 4.1, but his proof is incorrect since it depends on Theorems 1 and 2 of [17]; see Remark 3.2(e).) We note, incidentally, that an affirmative answer to Question 4.1 implies Case (4) of Theorem 3.1. This follows from the easily verified fact that, for any space $\langle X, \mathcal{T}\rangle$, $\langle X, \mathcal{T}(\kappa)\rangle$ is homeomorphic to the diagonal of $\kappa - \Box_{\xi\in\kappa} X_\xi$, where $X_\xi = \langle X, \mathcal{T}\rangle$ for all $\xi \in \kappa$.

We show in Theorem 4.4 that the answer to Question 4.1 is "yes" if each X_i is zero-dimensional. (In view of this and Remark 4.2(c), a negative answer to Question 4.1 would be highly unexpected.)

We first prove the following rather technical lemma (which will be used for both Theorem 4.4 and 4.6):

4.3. Lemma. *Let $\langle X_i \colon i \in I\rangle$ be a family of α-compact spaces, let $\omega \leqslant \kappa \leqslant m(\alpha)$ and $X = \kappa - \Box_{i\in I} X_i$, and assume that $S \subset X$ satisfies the following condition: For every zero-set Z of S and every $x \in X - Z$, there exists $J \subset I$ with $0 \neq |J| < \kappa$, and, for each $i \in J$, there exists an open set U_i in X_i such that $x \in \bigcap_{i\in J} \pi_i^{-1}(U_i) \subset X - Z$ and such that $\pi_i^{-1}(U_i) \cap S$ is clopen in S for every $i \in J$. Then S is α-compact.*

Proof. Since S is clearly closed in X, we may assume that $\alpha > \omega$. Let F be a z-ultrafilter on S with the α-intersection property. Then for each $i \in I$, $(\pi_i|S)^{\#}(\mathscr{F})$ is a z-ultrafilter on X_i with the α-intersection property (see Proposition 2.1), so there exists $x_i \in \bigcap (\pi_i|S)^{\#}(\mathscr{F})$. Let $x = \langle x_i \colon i \in I\rangle$. It will suffice to show that $x \in \bigcap \mathscr{F}$. Suppose, on the contrary, that $x \notin Z$ for some $Z \in \mathscr{F}$, and choose J and $\langle U_i \colon i \in J\rangle$ as in the statement of Lemma 4.3. Note that for each $i \in J$ there is a zero-set Z_i in X_i such that $x_i \in Z_i \subset U_i$. Then $Z_i \in (\pi_i|S)^{\#}(\mathscr{F})$, and hence $(\pi_i|S)^{-1}(Z_i) = \pi_i^{-1}(Z_i) \cap S$ is in $\mathscr{F}$, from which it follows that $\pi_i^{-1}(U_i) \cap S \in \mathscr{F}$. Moreover, it is clear that $Q = S - \bigcap_{i\in J} (\pi_i^{-1}(U_i) \cap S)$ is clopen in S and $Z \subset Q$, so $Q \in \mathscr{F}$.

Now let Φ be as described in Lemma 2.2 (but with X replaced by S) and let $\mathscr{B} = \{\pi_i^{-1}(U_i) \cap Q \colon i \in J\}$. It is easy to verify that $\mathscr{B} \in \Phi$, and thus $m = \min\{|\mathscr{A}| \colon \mathscr{A} \in \Phi\}$ is measurable by Lemma 2.2. Since clearly $\alpha \leqslant m$, we then have $m(\alpha) \leqslant m \leqslant |\mathscr{B}| \leqslant |J| < \kappa$, a contradiction. □

4.4. Theorem. *If $\langle X_i \colon i \in I\rangle$ is a family of zero-dimensional α-compact spaces, and if $\omega \leqslant \kappa \leqslant m(\alpha)$, then $\kappa - \Box_{i\in I} X_i$ is α-compact.*

Proof. We need only take $S = \kappa - \Box_{i\in I} X_i$ in Lemma 4.3. □

We denote the absolute of a space X by $E(X)$. Recall that $E(X)$ is zero-dimensional and that there is a perfect (and irreducible) map from $E(X)$ onto X. (See [21] for these and other properties of $E(X)$.)

4.5. Corollary. *If $\langle X_i \colon i \in I\rangle$ is a family of α-compact spaces, and if $\omega \leqslant \kappa \leqslant m(\alpha)$, then $\kappa - \Box_{i\in I} E(X_i)$ is α-compact.*

Proof. In view of Theorem 4.4, we need only note that a perfect pre-image of an α-compact space is α-compact [5, 4.4]. □

We note that the bound $m(\alpha)$ on κ in Theorem 4.4 is best possible. In fact, it is easy to see that if $\kappa > m(\alpha)$, $|I| \geqslant m(\alpha)$, and $|X_i| \geqslant 2$ for every $i \in I$, then $X = \kappa - \Box_{i\in I} X_i$ has a closed discrete subset D with $|D| = 2^{m(\alpha)}$. But then D is not α-compact, and hence X is not α-compact (see [5, 2.2(a) and 5.3]).

On the other hand, as an immediate consequence of Lemma 4.3 we have the following result:

4.6. Theorem. *If* $\langle X_i: i \in I\rangle$ *is a family of* α*-compact spaces, and if* $\omega \leqslant \kappa \leqslant m(\alpha)$, *then every closed discrete subspace of* $\kappa - \Box_{i\in I} X_i$ *is* α*-compact.*

We do not know whether "discrete" can be replaced by "zero-dimensional" in the statement of Theorem 4.6.

We call a space X *completely uniformizable* if X admits a compatible complete uniformity (see [9, 15.7]). (Equivalently, X is completely uniformizable if X is homeomorphic to a closed subspace of a product of metrizable spaces [6]. Note that the property "completely uniformizable" is called "Dieudonné-complete" in [7, 8.5.13] and "topologically complete" in [14, 6L].) It is known that a completely uniformizable space X is α-compact if and only if every closed discrete subspace of X is α-compact [16, 2.15 and 2.4]. Moreover, any κ-box product of completely uniformizable spaces is again completely uniformizable (see [13, 4.2(2)] or [20, 2.5(2)]). We thus have the following corollary of Theorem 4.6.

4.7. Corollary. *If* $\langle X_i: i \in I\rangle$ *is a family of completely uniformizable* α*-compact spaces, and if* $\omega \leqslant \kappa \leqslant m(\alpha)$, *then* $\kappa - \Box_{i\in I} X_i$ *is* α*-compact.*

4.8. Remarks. (a) Since every ω_1-compact (=realcompact) space is completely uniformizable [9, 15.14(a)], the hypothesis "completely uniformizable" in Corollary 4.7 is redundant for the case $\alpha = \omega_1$ (i.e., for the case described in Remark 4.2(c)).

(b) Let us say that a uniformity $\mathcal{U}$ on a set X is *α-complete* if every $\mathcal{U}$-Cauchy filter on X with the α-intersection property is convergent, and that a space X is *α-completely uniformizable* if X admits a compatible α-complete uniformity. Then, by adapting standard proofs, one can show that in Corollary 4.7 the hypothesis "completely uniformizable" can be replaced by "α-completely uniformizable".

Finally, we observe that Question 4.1 can be reduced to the apparently simpler Questions 4.9(1) and (2) below.

For any infinite cardinal λ, denote by $\mathbf{0}$ the constant function in the Tychonoff product $[0, 1]^\lambda$ with value 0. (In 4.9(2), $\prod$ indicates the Tychonoff product.)

4.9. Questions. (1) If $|I| < m(\mu)$ and if $X_i = [0, 1]^\mu - \{\mathbf{0}\}$ for every $i \in I$, does it follow that $\Box_{i\in I} X_i$ is μ^+-compact?

(2) If α is a limit cardinal, if $|I| < m(\alpha)$, and if $X_i = \prod_{\lambda<\alpha} ([0, 1]^\lambda - \{\mathbf{0}\})$ for every $i \in I$, does it follow that $\Box_{i\in I} X_i$ is α-compact?

4.10. Proposition. *If Questions* 4.9(1) *and* (2) *both have affirmative answers, then so does* 4.1.

Proof. Hušek has shown that if $\alpha = \mu^+$ (resp. α is a limit cardinal), then every α-compact space is homeomorphic to a closed subspace of some Tychonoff product of copies of $[0, 1]^{\mu} - \{\mathbf{0}\}$ (resp. copies of $\prod_{\lambda<\alpha}([0, 1]^{\lambda} - \{\mathbf{0}\})$) (see [12, Theorems 1 and 2]). The result is therefore an immediate consequence of a theorem of Kato [13, 4.1]. □

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References

[1] R.L. Blair, On υ-embedded sets in topological spaces, in: TOPO 72—General Topology and its Applications, Second Pittsburgh International Conference, December 18-22, 1972, Lecture Notes Math. 378 (Springer, Berlin/Heidelberg/New York, 1974) 46-79.

[2] R.L. Blair, Filter characterizations of z-, C^*-, and C-embeddings, Fund. Math. 90 (1976) 285-300.

[3] R.L. Blair, L.T. Polkowski and M.A. Swardson, Non-zero-dimensional topologies between zero-dimensional ones, Topology Proc. 12 (1987) 39-46.

[4] W.W. Comfort and S. Negrepontis, The theory of ultrafilters, Die Grundlehren der math. Wissenschaften, Band 211 (Springer, New York/Heidelberg/Berlin, 1974).

[5] W.W. Comfort and T. Retta, Generalized perfect maps and a theorem of I. Juhász, in: Rings of Continuous Functions, Lecture Notes Pure Appl. Math. 95 (Marcel Dekker, New York/Basel, 1985) 79-102.

[6] J. Dieudonné, Sur les espaces uniformes complets, Ann. Sci. École Norm. Sup. 56 (1939) 277-291.

[7] R. Engelking, General Topology (English trans.) (PWN, Warsaw, 1977).

[8] Z. Frolík, Realcompactness is a Baire-measurable property, Bull. Acad. Polon. Sér. Sci. Math. Astr. Phys. 19 (1971) 617-621.

[9] L. Gillman and M. Jerison, Rings of continuous functions, The University Series in Higher Math. (Van Nostrand, Princeton, 1960); reprinted (Springer-Verlag, New York/Heidelberg/Berlin, 1976).

[10] S. Hernández, Extension of continuous functions into uniform spaces, Proc. Amer. Math. Soc. 97 (1986) 355-360.

[11] H. Herrlich, Fortsetzbarkeit stetiger Abbildungen und Kompaktheitsgrad topologischer Räume, Math. Zeit. 96 (1967) 64-72.

[12] M. Hušek, The class of k-compact spaces is simple, Math. Zeit. 110 (1969) 123-126.

[13] A. Kato, Realcompactness of box products, Mem. Defense Acad. Japan 19 (1979) 1-4.

[14] J.L. Kelley, General Topology (Van Nostrand, New York, 1955).

[15] A.K. Misra, A topological view of P-spaces, Gen. Topology Appl. 2 (1972) 349-362.

[16] T. Retta, Topics in α-compactness and density character, Ph.D. thesis, Wesleyan University, Middletown, Connecticut, 1977.

[17] T. Retta, A note on α-compact spaces, Proc. Amer. Math. Soc. 89 (1983) 314-316.

[18] R.F. Wheeler, On separable z-filters, Gen. Topology Appl. 5 (1975) 333-345.

[19] S.W. Williams, Box product, in: K. Kunen and J. Vaughan, Eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam/New York/Oxford, 1984) 169-200.

[20] S.W. Williams, More realcompact spaces, Rings of Continuous Functions, Lecture Notes Pure Appl. Math. 95 (Marcel Dekker, New York/Basel, 1985) 289-300.

[21] R.G. Woods, A survey of absolutes of topological spaces, Topological Structures II, Part 2, Proceedings of the Symposium in Amsterdam, October 31-November 2, 1978, Math. Centre Tracts 116, Math. Centrum, Amsterdam, 1979, pp. 324-362.